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LETTER TO THE EDITOR

Hannay angle for a classical spinning particle in a rotating magnetic field

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Abstract. For the spin-1 system in a slowly varying magnetic field, we investigate its classical analogue to calculate the Hannay angle and explicitly check that this is related to Berry's phase according to the well known semiclassical formula.

Phases in quantum mechanics have a long and distinguished history. One such phase factor which has drawn great attention recently is Berry's phase (Berry 1984).

For the quantal Hamiltonian which depends on slowly varying external parameters, Berry showed that, in the adiabatic approximation, the solution of the time-dependent Schrodinger equation initially chosen to be a non-degenerate eigenstate of the instantaneous Hamiltonian acquires, in addition to the usual 'dynamical phase factor', a geometrical phase $\gamma_n(C)$ as the parameters are slowly varied along a closed circuit C in the parameter space in time T. Here n reflects the dependence of the phase on the quantum numbers of the eigenstate. This perceptive observation of Berry also has a classical analogue, called the Hannay angle (Berry 1985, Hannay 1985). As one goes round a closed curve C in the parameter space of a classical Hamiltonian this induces, in the adiatic approximation, a shift $\Delta\theta(C, I)$ in the angle variable θ conjugate to the action variable I. Berry and Hannay have further shown that this classical shift is the action derivative of the semiclassical Berry phase, i.e.

$$\Delta\theta(C, I) = -\hbar\partial\gamma_n(C)/\partial I = -\partial\gamma_n(C)/\partial n. \tag{1}$$

This semiclassical formula has been verified in a number of examples considered by Berry (1985) himself and others.

The prototype Hamiltonian much investigated in the literature for Berry's phase calculation is (Berry 1984, Kuratsuji and Iida 1985):

$$H = -\boldsymbol{J}\cdot\boldsymbol{B}. \tag{2}$$

This is the Hamiltonian for spin J in a magnetic field $\boldsymbol{B}$ whose direction is varied slowly. A classical analogue of (2) discussed by Gozzi and Thacker (1987) uses Grassmannian variables which, however, limits their analysis to the spin-$\frac{1}{2}$ representation. We thought the case of higher spin would also be worth investigating. Here we consider such a classical dynamics, which corresponds to the Hamiltonian in (2) for spin-1 representation, and calculate the Hannay angle to check whether this is compatible with the semiclassical formula in (1).

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The classical Hamiltonian we consider describes the motion of a magnetic moment in a time-varying magnetic field. We use ordinary classical variables as opposed to those used by Gozzi and Thacker. Some differences from their work are commented on at the end.

We proceed with the elementary computation of the classical phase shift(s) directly from the familiar equation of motion (Slichter 1978)

$$\dot{S}_k = \varepsilon_{klm} S_l B_m. \tag{3}$$

This is identical to the Schrodinger equation of a spin-1 particle in the same field (Series 1978)

$$\mathrm{i}\hbar\dot{\psi}_k = H_{kl}\psi_l$$

where the Hamiltonian $-(J_m)_{kl}B_m$ has a form similar to (2), and is given in terms of the angular momentum operators (with respect to some basis)

$$(J_m)_{kl} = -\mathrm{i}\hbar\varepsilon_{klm}, \tag{4}$$

which constitute the regular representation of angular momentum algebra. Hence we can say that (3) is the classical analogue of spin-1 dynamics. The exact expression for Berry's phase being known in this case (Berry 1984), our aim is to verify that, in accordance with (1), the classical shift equals the magnetic quantum number derivative of the quantum phase.

Let us first rewrite (3) in the matrix form

$$\dot{S}_k = \hat{B}_{kl} S_l \tag{5}$$

where $\hat{B}$ is an antisymmetric matrix associated with the 'Euler vector' $\boldsymbol{B}$ of magnitude

$$B = (B_1^2 + B_2^2 + B_3^2)^{1/2}. \tag{6}$$

The most obvious way to solve (5) is to look for a matrix U which diagonalises $\hat{B}$, i.e.

$$U^{-1}\hat{B}U = \hat{B}_{\mathrm{d}} \tag{7}$$

$\hat{B}_{\mathrm{d}}$ being the diagonal eigenvalue matrix with entries 0, $\pm\mathrm{i}B$. Let us introduce the 'new' variables

$$\tilde{S}_k = U^{\dagger}_{kl} S_l = U^{-1}_{kl} S_l \tag{8}$$

(where $\dagger$ denotes Hermitian conjugation); in terms of these the equations of motion become

$$\dot{\tilde{S}}_k = \hat{B}_{\mathrm{d}kl}\tilde{S}_l - (U^{+}\dot{U})_{kl}\tilde{S}_l. \tag{9}$$

Note that since the external parameters B vary with time, U depends on time explicitly through them and therefore the second term on the right-hand side of (9) is in general non-vanishing. However, this term can be ignored instantaneously if the parameters vary slow enough and it is natural in the adiabatic approximation to look for a solution to (9) of the type

$$S_k = \varphi_k(t)\exp(-\mathrm{i}\theta_k(t)). \tag{10}$$

The Hannay angle phenomenon which interests us appears in the angle variables $\theta_k(t)$, defined by (10). To see this we substitute (10) into (9) and then separate the real and imaginary parts, which yields

$$\dot{\varphi}_k = -\sum_{k\neq l} \operatorname{Re}\hat{\varkappa}_{kl}\varphi_l \qquad k, l = 1\text{–}3 \tag{11a}$$

and

$$\dot{\theta}_k = -\sum_{k\neq l} \operatorname{Im} \hat{\varkappa}_{kl}\phi_1 + \mathrm{i}\,\hat{B}_{\mathrm{d}kk} - \mathrm{i}(U+\dot{U})_{kk} \qquad k, l = 1\text{-}3. \tag{11b}$$

Here we have introduced the anti-Hermitian matrix:

$$\hat{\varkappa}_{kl} = \exp(\mathrm{i}\theta_k)(U^{\dagger}\dot{U})_{kl}\exp(-\mathrm{i}\theta_1). \tag{12}$$

In the spirit of an adiabatic analysis, one would average over one period of the fast motion. Following Berry, one can then replace time averages by angle averages over the true fast angle variables. As one of the eigenvalues is zero, one variable, say θ_1, does not change at all. Therefore, strictly speaking, while considering the angular average, the θ_1 average has to disappear, leaving us with the definition

$$\langle\ldots\rangle = \int_0^{\pi}\frac{\mathrm{d}\theta_2}{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\theta_3}{2\pi}\ldots. \tag{13}$$

Using this averaging procedure, the off-diagonal elements of $\langle\hat{\varkappa}\rangle$ are found to vanish; of interest are surviving diagonal elements. Consequently, this implies from (11a) that the (angle averaged) φ_k are adiabatic invariants, and also that the net change in (angle averaged) θ_k, after a round trip in the parameter space during time T, is given by

$$\langle\theta_k(T)-\theta_k(0)\rangle = \mathrm{i}\int_0^T \mathrm{d}t\, B_{\mathrm{d}kk} - \mathrm{i}\int_0^T \mathrm{d}t\,(U^{\dagger}\dot{U})_{kk} \qquad k=1\text{-}3. \tag{14}$$

The interpretation of both the terms on the right-hand side of (14) is quite clear: the first term is the dynamical contribution accumulated during time T, and the second term is the Hannay angle:

$$\begin{aligned}\Delta\theta_k(C, I_k) &= \int_0^T \mathrm{d}t\,\operatorname{Im}(U^{\dagger}\dot{U})_{kk}\\ &= \oint_C \mathrm{d}B_p\,\operatorname{Im}\left(U^{\dagger}\frac{\partial U}{\partial B_p}\right)_{kk}\\ &= \int \mathrm{d}S_m\,\operatorname{Im}\left(\varepsilon_{mnp}\frac{\partial\hat{U}^{\dagger}}{\partial B_n}\frac{\partial\hat{U}}{\partial B_p}\right)_{kk}\\ &= \int \mathrm{d}S_m\,\operatorname{Im}\left(\varepsilon_{mnp}\frac{\partial X_l^{(k)*}}{\partial B_n}\frac{\partial X_l^{(k)}}{\partial B_p}\right)\end{aligned} \tag{15a-d}$$

where (15c) follows from (15b) via Stokes' theorem, and in (15d) we have introduced the following notation:

$$\begin{aligned}X_l^{(1)} &= B_l/B\\ X_l^{(2)} &= X_l^{(3)*} = \frac{B_lB_3 + \mathrm{i}\varepsilon_{lm3}B_mB - \delta_{3l}B^2}{B\sqrt{2}(B_1^2+B_2^2)^{1/2}}.\end{aligned} \tag{16}$$

$X^{(k)}$ denote the three three-component (column) eigenvectors of $\hat{B}$ with eigenvalues 0, $-\mathrm{i}B$ and $\mathrm{i}B$, respectively. A straightforward, though tedious, calculation shows that

$$\begin{aligned}\Delta\theta_1(C, I_1) &= 0\\ \Delta\theta_2(C, I_2) &= -\Delta\theta_3(C, I_3) = -\Omega(C).\end{aligned} \tag{17}$$

Here $\Omega(c)$ is the solid angle subtended by C at the origin of the parameter space. The result given by (17) fulfils our expectation from (1). Replacing the derivative by the ratio of finite differences, we obtain

$$-\frac{\Delta\gamma_n(C)}{\Delta n}=-\frac{\gamma_1(C)-\gamma_0(C)}{1-0}=-\Omega(C)=\Delta\theta_2(C, I_2).$$

We have used here the exact result due to Berry (1984) that $\gamma_n(C)=n\Omega(C)$ for the Hamiltonian of (2).

Similar phase shift expressions were derived by Gozzi and Thacker (1987) for their spin-$\frac{1}{2}$ Grassmannian system. They compute the 'angle 2-form' (Berry 1984) in their model whose action derivative yields the classical shifts similar to our (17). Their averaging procedure involves averaging over all the three angles, whereas in our case it would be wrong to average over θ_1, which remains fixed. The expressions for the Lagrangian and Hamiltonian in our spin-1 model which, however, we had no occasion to exploit, differ from their spin-$\frac{1}{2}$ model. For example, in terms of the action-angle variables I_k, θ_k, $k=1$-3, our Hamiltonian is given as $B(I_2+I_3)$. In contrast, in their case $H=B(I_2-I_3)$.

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